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THE NEUMANN PROBLEM FOR THE ∞-LAPLACIAN

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ABSTRACT. We survey the results of the paper [GMPR] related to to the theory of viscosity solutions of the ∞-Laplacian with Neuman boundary conditions. We study the limit as $p \to \infty$ of solutions of $-\Delta_p u_p = 0$ in a domain Ω with $|Du_p|^{p-2}\partial u_p/\partial\nu = g$ on $\partial\Omega$. We obtain a natural minimization problem that is verified by a limit point of $\{u_p\}$ and a limit problem that is satisfied in the viscosity sense. It turns out that the limit variational problem is related to the Monge-Kantorovich mass transfer problem when the measures are supported on $\partial\Omega$.

1. INTRODUCTION.

In this survey we study the natural Neumann boundary conditions that appear when one considers the ∞-Laplacian in a smooth bounded domain as limit of the Neumann problem for the p-Laplacian as $p \to \infty$.

Let $\Delta_p u = \operatorname{div}\left(|Du|^{p-2}Du\right)$ be the p-Laplacian. The ∞-Laplacian is the limit operator $\Delta_\infty = \lim_{p\to\infty} \Delta_p$ given by

$$\Delta_\infty u = \sum_{i,j=1}^{N} \frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_j \partial x_i}\frac{\partial u}{\partial x_i}$$

in the viscosity sense. A fundamental result of Jensen [J] establishes that the Dirichlet problem for Δ_∞ is well posed in the viscosity sense.

When considering the Neumann problem, boundary conditions that involve the outer normal derivative, $\partial u/\partial\nu$ have been addressed from the point of view of viscosity solutions for fully nonlinear equations in [B] and [ILi]. In these references it is proved that there exist viscosity solutions and comparison principles between them when appropriate hypothesis are satisfied. In particular strict monotonicity relative to the solution u is needed, a property that homogeneous equations do not satisfy.

We study the Neumann problem for the ∞-Laplacian obtained as the limit as $p \to \infty$ of the problems

$$\begin{cases} -\Delta_p u = 0 & \text{in } \Omega, \\ |Du|^{p-2}\frac{\partial u}{\partial\nu} = g & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary and $\frac{\partial}{\partial\nu}$ is the outer normal derivative. The boundary data g is a continuous function that necessarily

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verifies the compatibility condition

$$\int_{\partial\Omega} g = 0,$$

otherwise there is no solution to (1.1). Imposing the normalization

$$\int_{\Omega} u = 0 \tag{1.2}$$

there exists a unique solution to problem (1.1) that we denote by u_p. By standard techniques this solution can also be obtained by a variational principle . In fact, we can write

$$\int_{\partial\Omega} u_p\, g = \max\left\{\int_{\partial\Omega} wg \colon w \in W^{1,p}(\Omega),\ \int_{\Omega} w = 0\,,\ \int_{\Omega} |Dw|^p \leq 1\right\}.$$

Our first result states that there exist limit points of u_p as $p \to \infty$ and that they are maximizers of a variational problem that is a natural limit of these variational problems.

Observe that for $q > N$ the set $\{u_p\}_{p>q}$ is bounded in $C^{1-p/q}(\overline{\Omega})$. Let v_∞ be a uniform limit of a subsequence $\{u_{p_i}\}$, $p_i \to \infty$.

Theorem 1.1. *A limit function v_∞ is a solution to the maximization problem*

$$\int_{\partial\Omega} v_\infty g = \max\left\{\int_{\partial\Omega} wg \colon w \in W^{1,\infty}(\Omega),\ \int_{\Omega} w = 0,\ \|Dw\|_\infty \leq 1\right\}. \tag{1.3}$$

An equivalent dual statement is the minimization problem

$$\|Dv_\infty\|_\infty = \min\left\{\|Dw\|_\infty \colon w \in W^{1,\infty}(\Omega),\ \int_{\Omega} w = 0,\ \int_{\partial\Omega} wg \geq 1\right\}. \tag{1.4}$$

The maximization problem (1.3) is also obtained by applying the Kantorovich optimality principle to a mass transfer problem for the measures $\mu^+ = g^+\mathcal{H}^{N-1}\llcorner\,\partial\Omega$ and $\mu^- = g^-\mathcal{H}^{N-1}\llcorner\,\partial\Omega$ that are concentrated on $\partial\Omega$. The mass transfer compatibility condition $\mu^+(\partial\Omega) = \mu^-(\partial\Omega)$ holds since g has zero average on $\partial\Omega$. The maximizers of (1.3) are called maximal Kantorovich potentials [Am].

Evans and Gangbo [EG] have considered mass transfer optimization problems between absolutely continuous measures that appear as limits of p-Laplacian problems. A very general approach is discussed in [BBP], where a problem related to but different from ours is discussed (see Remark 4.3 in [BBP].)

Our next results discusses the equation that v_∞ satisfies in the viscosity sense.

Theorem 1.2. *A limit v_∞ is a solution of*

$$\begin{cases} \Delta_\infty u = 0 & \text{in } \Omega, \\ B(x,u,Du) = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

in the viscosity sense. Here

$$B(x,u,Du) \equiv \begin{cases} \min\left\{|Du| - 1\ ,\ \frac{\partial u}{\partial\nu}\right\} & \text{if } g(x) > 0, \\ \max\{1 - |Du|\ ,\ \frac{\partial u}{\partial\nu}\} & \text{if } g(x) < 0, \\ H(|Du|)\frac{\partial u}{\partial\nu} & \text{if } g(x) = 0, \\ \frac{\partial u}{\partial\nu} = 0 & \text{if } x \in \{g(x) = 0\}^o, \end{cases}$$

and $H(a)$ is given by

$$H(a) = \begin{cases} 1 & \text{if } a \geq 1, \\ 0 & \text{if } 0 \leq a < 1. \end{cases}$$

Notice that the boundary condition only depends on the sign of g. The question we wish to address is whether we have uniqueness of viscosity solutions of (1.5). Unfortunately this is not the case as it will be shown by an example discussed in Section §3. Nevertheless we can say something about uniqueness of v_∞ under some favorable geometric assumptions on g and Ω by adapting techniques from [EG]. See [GMPR] for details.

2. The Neumann problem

In this section we prove that there exists a limit, v_∞, of the solutions at level p, u_p. It satisfies a variational principle (1.3) and it is a solution to (1.5).

Recall from the introduction that we call u_p the solution of (1.1) with the normalization (1.2). As we have mentioned, this solution can be obtained by a variational principle. Indeed, consider the minimum in S of the following functional

$$J_p(u) = \int_\Omega |Du|^p - \int_{\partial\Omega} ug$$

where S is given by

$$S = \left\{ u \in W^{1,p}(\Omega) \ : \ \int_\Omega u = 0 \right\}.$$

It follows from standard techniques that the functional J_p attains a unique minimun in S. We shall need an alternative variational formulation that is equivalent to the previous one

$$M_p = \max\left\{ \int_{\partial\Omega} wg \ : \ w \in W^{1,p}(\Omega) \ : \ \int_\Omega w = 0\,,\ \int_\Omega |Dw|^p \leq 1 \right\}.$$

Denoting a maximizer by $\tilde{u}_p$ we have

$$\Delta_p \tilde{u}_p = 0$$

with the boundary condition

$$|D\tilde{u}_p|^{p-2}\frac{\partial \tilde{u}_p}{\partial \nu} = \frac{g}{M_p}.$$

Hence, it holds

$$u_p \equiv M_p^{1/(p-1)} \tilde{u}_p.$$

A key point is to observe that the quantity M_p is uniformly bounded in $p \in [2,\infty)$. To see this fact we use the trace inequality to obtain

$$M_p = \int_{\partial\Omega} \tilde{u}_p g \leq \|g\|_\infty \int_{\partial\Omega} |\tilde{u}_p| \leq C_1 \|g\|_\infty \int_\Omega |D\tilde{u}_p| \leq C_1 \|g\|_\infty.$$

Suppose that we have a sequence $\{u_p\}$ of solutions to (1.1). We derive some estimates on the family u_p. Since we are interested in large values of p we may assume that $p > N$ and hence $u_p \in C^\alpha(\overline{\Omega})$. Multiplying the equation by u_p and integrating we obtain,

$$\int_\Omega |Du_p|^p = \int_{\partial\Omega} u_p\, g \leq \left(\int_{\partial\Omega} |u_p|^p \right)^{1/p} \left(\int_{\partial\Omega} |g|^{p'} \right)^{1/p'} \tag{2.1}$$

where p' is the exponent conjugate to p, that is $1/p'+1/p=1$. Recall the following trace inequality, see for example [E],

$$\int_{\partial\Omega} |\phi|^p d\sigma \leq Cp \left(\int_\Omega |\phi|^p + |D\phi|^p dx\right),$$

where C is a constant that does not depend on p. Going back to (2.1), we get,

$$\int_\Omega |Du_p|^p \leq \left(\int_{\partial\Omega} |g|^{p'}\right)^{1/p'} C^{1/p} p^{1/p} \left(\int_\Omega |u_p|^p + |Du_p|^p dx\right)^{1/p}.$$

On the other hand, for large p we have

$$|u_p(x) - u_p(y)| \leq C_p |x-y|^{1-\frac{N}{p}} \left(\int_\Omega |Du_p|^p dx\right)^{1/p}.$$

Since we are assuming that $\int_\Omega u_p = 0$, we may choose a point y such that $u_p(y)=0$, and hence

$$|u_p(x)| \leq C(p,\Omega) \left(\int_\Omega |Du_p|^p dx\right)^{1/p}.$$

The arguments in [E], pages 266-267, show that **the constant $C(p,\Omega)$ can be chosen uniformly in p.** Hence, we obtain

$$\int_\Omega |Du_p|^p \leq \left(\int_{\partial\Omega} |g|^{p'}\right)^{1/p'} C^{1/p} p^{1/p} (C_2^p + 1)^{1/p} \left(\int_\Omega |Du_p|^p dx\right)^{1/p}.$$

Taking into account that $p' = p/(p-1)$, for large values of p we get

$$\left(\int_\Omega |Du_p|^p\right)^{1/p} \leq \alpha_p \left(\int_{\partial\Omega} |g|^{p'}\right)^{1/p'}$$

where $\alpha_p \to 1$ as $p \to \infty$. Next, fix m, and take $p > m$. We have,

$$\left(\int_\Omega |Du_p|^m\right)^{1/m} \leq |\Omega|^{\frac{1}{m}-\frac{1}{p}} \left(\int_\Omega |Du_p|^p\right)^{1/p} \leq |\Omega|^{\frac{1}{m}-\frac{1}{p}} \left(\int_{\partial\Omega} |g|^{p'}\right)^{1/p},$$

where $|\Omega|^{\frac{1}{m}-\frac{1}{p}} \to |\Omega|^{\frac{1}{m}}$ as $p \to \infty$. Hence, there exists a weak limit in $W^{1,m}(\Omega)$ that we will denote by v_∞. This weak limit has to verify

$$\left(\int_\Omega |Dv_\infty|^m\right)^{1/m} \leq |\Omega|^{\frac{1}{m}}.$$

As the above inequality holds for every m, we get that $v_\infty \in W^{1,\infty}(\Omega)$ and moreover, taking the limit $m \to \infty$,

$$|Dv_\infty| \leq 1, \qquad \text{a.e. } x \in \Omega.$$

Lemma 2.1. *The subsequence u_{p_i} converges to v_∞ uniformly in $\overline{\Omega}$.*

Proof. From our previous estimates we know that

$$\left(\int_\Omega |Du_p|^p dx\right)^{1/p} \leq C,$$

uniformly in p. Therefore we conclude that u_p is bounded (independently of p) and has a uniform modulus of continuity. Hence u_p converges uniformly to v_∞. □

Proof of Theorem 1.1. Multiplying by u_p, passing to the limit, and using Lemma 2.1, we obtain,

$$\lim_{p\to\infty}\int_\Omega |Du_p|^p = \lim_{p\to\infty}\int_{\partial\Omega} u_p g = \int_{\partial\Omega} v_\infty g.$$

If we multiply (1.1) by a test function w, we have, for large enough p,

$$\begin{aligned}\int_{\partial\Omega} wg &\le \left(\int_\Omega |Du_p|^p\right)^{(p-1)/p}\left(\int_\Omega |Dw|^p\right)^{1/p}\\ &\le \left(\int_{\partial\Omega} v_\infty g d\sigma + \delta\right)^{(p-1)/p}\left(\int_\Omega |Dw|^p\right)^{1/p}.\end{aligned}$$

As the previous inequality holds for every $\delta > 0$, passing to the limit as $p \to \infty$ we conclude,

$$\int_{\partial\Omega} wg \le \left(\int_{\partial\Omega} v_\infty g\right)\|Dw\|_\infty.$$

Hence, the function v_∞ verifies,

$$\int_{\partial\Omega} v_\infty g = \max\left\{\int_{\partial\Omega} wg \;:\; w\in W^{1,\infty}(\Omega),\ \int_\Omega w = 0,\ \|Dw\|_\infty \le 1\right\},$$

or equivalently,

$$\|Dv_\infty\|_\infty = \min\left\{\|Dw\|_\infty : w\in W^{1,\infty}(\Omega),\ \int_\Omega w = 0,\ \int_{\partial\Omega} wg \le 1\right\}.$$

□

Following [B] let us recall the definition of viscosity solution taking into account general boundary conditions for elliptic problems. Assume

$$F : \overline{\Omega}\times\mathbb{R}^N\times\mathbb{S}^{N\times N}\to\mathbb{R}$$

a continuous function. The associated equation

$$F(x, \nabla u, D^2u) = 0$$

is called (degenerate) elliptic if

$$F(x,\xi,X) \le F(x,\xi,Y) \quad \text{if } X \ge Y.$$

Definition 2.1. *Consider the boundary value problem*

$$\left\{\begin{array}{ll} F(x, Du, D^2u) = 0 & \text{in } \Omega,\\ B(x, u, Du) = 0 & \text{on } \partial\Omega.\end{array}\right. \tag{2.2}$$

(1) *A lower semi-continuous function u is a viscosity supersolution if for every $\phi\in C^2(\overline{\Omega})$ such that $u-\phi$ has a strict minimum at the point $x_0\in\overline{\Omega}$ with $u(x_0)=\phi(x_0)$ we have: If $x_0\in\partial\Omega$ the inequality*

$$\max\{B(x_0,\phi(x_0),D\phi(x_0)),\ F(x_0,D\phi(x_0),D^2\phi(x_0))\} \ge 0$$

holds, and if $x_0\in\Omega$ then we require

$$F(x_0,D\phi(x_0),D^2\phi(x_0)) \ge 0.$$

(2) *An upper semi-continuous function u is a subsolution if for every $\phi \in C^2(\overline{\Omega})$ such that $u-\phi$ has a strict maximum at the point $x_0 \in \overline{\Omega}$ with $u(x_0) = \phi(x_0)$ we have: If $x_0 \in \partial\Omega$ the inequality*

$$\min\{B(x_0, \phi(x_0), D\phi(x_0)),\ F(x_0, D\phi(x_0), D^2\phi(x_0))\} \leq 0$$

holds, and if $x_0 \in \Omega$ then we require

$$F(x_0, D\phi(x_0), D^2\phi(x_0)) \leq 0.$$

(3) *Finally, u is a viscosity solution if it is a super and a subsolution.*

We will use the following notation

$$F_p(\eta, X) \equiv -Trace(A_p(\eta)X),$$

where

$$A_p(\eta) = Id + (p-2)\frac{\eta \otimes \eta}{|\eta|^2}, \quad \text{if } \eta \neq 0, \qquad A_p(0) = I_N,$$

and the notation

$$B_p(x, u, \eta) \equiv |\eta|^{p-2} < \eta, \nu(x) > -g(x). \tag{2.3}$$

It is not difficult to see that continuous (in $\overline{\Omega}$) weak solutions of (1.1) are indeed viscosity solutions.

Lemma 2.2. *Let u be a continuous weak solution of* (1.1) *for $p > 2$. Then u is a viscosity solution of*

$$\begin{cases} F_p(Du, D^2u) = 0 & \text{in } \Omega, \\ B_p(x, u, Du) = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

Proof. For points $x_0 \in \Omega$ and test functions ϕ such that $u(x_0) = \phi(x_0)$ and $u - \phi$ has a strict minimum at x_0 the argument is a simple variation of the argument in [JLM].

If $x_0 \in \partial\Omega$ we want to prove

$$\begin{aligned} \max\{\, &|D\phi(x_0)|^{p-2} < D\phi(x_0), \nu(x_0) > -g(x_0), \\ &-(p-2)|D\phi|^{p-4}\Delta_\infty\phi(x_0) - |D\phi|^{p-2}\Delta\phi(x_0)\} \geq 0. \end{aligned}$$

Assume that this is not the case. Multiplying by $(\psi - u)^+$ extended to zero outside of $B(x_0, r)$ we obtain

$$\int_{\{\psi>u\}} |D\psi|^{p-2}D\psi D(\psi - u) < \int_{\partial\Omega\cap\{\psi>u\}} g(\psi - u),$$

and

$$\int_{\{\psi>u\}} |Du|^{p-2}DuD(\psi - u) \geq \int_{\partial\Omega\cap\{\psi>u\}} g(\psi - u).$$

Therefore,

$$\begin{aligned} C(N,p) &\int_{\{\psi>u\}} |D\psi - Du|^p \\ &\leq \int_{\{\psi>u\}} \langle |D\psi|^{p-2}D\psi - |Du|^{p-2}Du, D(\psi - u)\rangle < 0, \end{aligned}$$

again a contradiction. This proves that u is a viscosity supersolution. The proof of the fact that u is a viscosity subsolution is similar. □

Remark 2.1. *If B_p is monotone in the variable $\frac{\partial u}{\partial \nu}$ Definition 2.1 takes a simpler form, see* [B]. *This is indeed the case for* (2.3). *More concretely, if u is a supersolution and $\phi \in C^2(\overline{\Omega})$ is such that $u - \phi$ has a strict minimum at x_0 with $u(x_0) = \phi(x_0)$, then*

(1) *if $x_0 \in \Omega$, then*

$$-\left\{\frac{|D\phi(x_0)|^2 \Delta\phi(x_0)}{p-2} + \Delta_\infty \phi(x_0)\right\} \geq 0,$$

and if

(2) *If $x_0 \in \partial\Omega$, then*

$$|D\phi(x_0)|^{p-2}\langle D\phi(x_0), \nu(x_0)\rangle \geq g(x_0).$$

Note however that (1.5) *does not verify this monotonicity condition.*

Proof of Theorem 1.2. (Sketch) First, note that $-\Delta_\infty u_\infty = 0$ in the viscosity sense in Ω by standard arguments (See [J] or [BBM].)

The point is to check the boundary condition. There are six cases to be considered.

Case 1: $v_\infty - \phi$ has a strict minimum at $x_0 \in \partial\Omega$ with $g(x_0) > 0$. Using the uniform convergence of u_{p_i} to v_∞ we obtain that $u_{p_i} - \phi$ has a minimum at some point $x_i \in \overline{\Omega}$ with $x_i \to x_0$. If $x_i \in \Omega$ for infinitely many i, we obtain

$$-\Delta_\infty \phi(x_0) \geq 0.$$

On the other hand if $x_i \in \partial\Omega$ we have, by Remark 2.1,

$$|D\phi|^{p_i-2}(x_i)\frac{\partial\phi}{\partial\nu}(x_i) \geq g(x_i).$$

Since $g(x_0) > 0$, we have $D\phi(x_0) \neq 0$, and we obtain

$$|D\phi|(x_0) \geq 1.$$

Moreover, we also have

$$\frac{\partial\phi}{\partial\nu}(x_0) \geq 0.$$

Hence, if $v_\infty - \phi$ has a strict minimum at $x_0 \in \partial\Omega$ with $g(x_0) > 0$, we have

$$\max\left\{\min\{-1 + |D\phi|(x_0), \frac{\partial\phi}{\partial\nu}(x_0)\}\ , -\Delta_\infty\phi(x_0)\right\} \geq 0. \tag{2.5}$$

Case 2: $v_\infty - \phi$ has a strict maximum at $x_0 \in \partial\Omega$ with $g(x_0) > 0$. The argument is similar to Case 1.

Case 3: $v_\infty - \phi$ has a strict maximum at x_0 with $g(x_0) < 0$. Using the uniform convergence of u_{p_i} to v_∞ we obtain that $u_{p_i} - \phi$ has a maximum at some point $x_i \in \overline{\Omega}$ with $x_i \to x_0$. If $x_i \in \Omega$ for infinitely many i, we can argue as before and obtain

$$-\Delta_\infty \phi(x_0) \leq 0.$$

On the other hand if $x_i \in \partial\Omega$ we have

$$|D\phi|^{p_i-2}(x_i)\frac{\partial\phi}{\partial\nu}(x_i) \leq g(x_i).$$

Since $g(x_0) < 0$, $D\phi(x_0) \neq 0$ and we obtain

$$|D\phi|(x_0) \geq 1,$$

and

$$\frac{\partial \phi}{\partial \nu}(x_0) \leq 0.$$

Hence, the following inequality holds

$$\min\left\{\max\{1-|D\phi|(x_0), \frac{\partial \phi}{\partial \nu}(x_0)\}\ , -\Delta_\infty \phi(x_0)\right\} \leq 0. \tag{2.6}$$

Case 4: $v_\infty - \phi$ has a strict minimum at $x_0 \in \partial\Omega$ with $g(x_0) < 0$. The argument is similar to Case 3.

Case 5: $v_\infty - \phi$ has a strict minimum at $x_0 \in \partial\Omega$ with $g(x_0) = 0$. Using the uniform convergence of u_{p_i} to v_∞ we obtain that $u_{p_i} - \phi$ has a minimum at some point $x_i \in \overline{\Omega}$ with $x_i \to x_0$. If $x_i \in \Omega$ for infinitely many i, we can argue as before and obtain

$$-\Delta_\infty \phi(x_0) \geq 0.$$

On the other hand if $x_i \in \partial\Omega$ we have

$$|D\phi|^{p_i-2}(x_i)\frac{\partial \phi}{\partial \nu}(x_i) \geq g(x_i).$$

If $D\phi(x_0) = 0$, then we have

$$\frac{\partial \phi}{\partial \nu}(x_0) = 0.$$

If $D\phi(x_0) \neq 0$ we obtain

$$\frac{\partial \phi}{\partial \nu}(x_i) \geq \left(\frac{1}{|D\phi|}(x_i)\right)^{p_i-2} g(x_i).$$

If $|D\phi(x_0)| \geq 1$ then we have

$$\frac{\partial \phi}{\partial \nu}(x_0) \geq 0.$$

Therefore, the following inequality holds

$$\max\left\{H(|D\phi|(x_0))\frac{\partial \phi}{\partial \nu}(x_0)\ , -\Delta_\infty \phi(x_0)\right\} \geq 0. \tag{2.7}$$

If x_0 belongs to the interior of the set $\{g = 0\}$ then we have,

$$|D\phi|^{p_i-2}(x_i)\frac{\partial \phi}{\partial \nu}(x_i) \geq g(x_i) = 0.$$

Hence, passing to the limit, we obtain

$$\frac{\partial \phi}{\partial \nu}(x_0) \geq 0.$$

Therefore

$$\max\left\{\frac{\partial \phi}{\partial \nu}(x_0)\ , -\Delta_\infty \phi(x_0)\right\} \geq 0. \tag{2.8}$$

Case 6: $v_\infty - \phi$ has a strict maximum at x_0 with $g(x_0) = 0$. The argument is similar to Case 5. □

Remark 2.2. *If u_p is the solution of* (1.1) *with boundary data g and $\hat{u}_p$ is the solution with boundary data $\hat{g} = \lambda g$, $\lambda > 0$, then*

$$u(x) = \lambda^{-1/(p-1)}\hat{u}(x).$$

Therefore the limit v_∞ is the same if we consider any positive multiple of g as boundary data and the same subsequence.

As a consequence the limit problem must be invariant by scalar multiplication of the data g. One could naively conjecture that the limits depends only on the sign of g, however this conjecture is not true as we will see in §3 below.

3. Examples

Example: An Interval. In $\Omega = (-L, L)$ with $g(L) = -g(-L) > 0$ the limit of the solutions of (1.1), u_p, turns out to be $u_\infty(x) = x$. It is easy to check that this function is indeed the unique solution of the maximization problem (1.3) and of the problem (1.5).

Example: The Annulus. Let Ω be the annulus

$$\Omega = \{r_1 < |x| < r_2\}.$$

Let us begin with a function g_0 that is a positive constant g_1 on $|x| = r_1$ and a negative constant g_2 on $|x| = r_2$ satisfyting the constraint

$$\int_{\partial\Omega} g_0 = \int_{|x|=r_1} g_1 + \int_{|x|=r_2} g_2 = 0.$$

As we stated in the introduction, the limit v_∞ is the cone,

$$v_\infty(x) = C(x) = \left(\frac{1}{|\Omega|}\int_\Omega |y|\right) - |x|. \tag{3.9}$$

To check this fact we observe that, by uniqueness, the solutions u_p of (1.1) are radial hence the limit v_∞ must be a radial function. Direct integration shows that it must be a cone with gradient one.

Note however that the cone (3.9) may not be a maximizer of (1.3) for another nonradial boundary datum g with $\text{sign}(g) = \text{sign}(g_0)$. In fact, consider a cone with the vertex slightly displaced,

$$C_{x_0}(x) = C - |x - x_0|. \tag{3.10}$$

One may concentrate g on $|x| = r_2$ near a point $\bar{x}$ and on $|x| = r_1$ near a point $\hat{x}$ preserving the total integral and the sign. It is easy to show that in this case the centered cone given by (3.9) does not maximizes (1.3) since for a suitable g we obtain

$$\int_{\partial\Omega} g(x)C(x)\,dx < \int_{\partial\Omega} g(x)C_{x_0}(x)\,dx.$$

Since this can be done without altering the sign of g we have that **there is no uniqueness for the limit problem** (1.5). Moreover, the limit v_∞ depends on the shape of g not only on its sign (see Remark 2.2.)

Example: The Disk. Now let us present a more interesting and non-trivial example of a domain and boundary data such that uniqueness holds. Let Ω be a disk in $\mathbb{R}^2$, $D = \{|(x,y)| < 1\}$ with boundary datum $g(x,y) > 0$ for $x > 0$ and

$g(x,y) < 0$ for $x < 0$ with $\int_{\partial D} g = 0$. In this case, by using arguments from the Monge-Kantorovich theory we have the uniqueness of the limit $\lim_{p\to\infty} u_p$. See [GMPR] for the details.

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